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Accurate computations for steep solitary waves

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Finite-amplitude solitary waves in water of arbitrary uniform depth are considered. A numerical scheme based on series truncation is presented to calculate the highest solitary wave. It is found that the ratio of the amplitude of the wave versus the depth is 0.833 22. This value is compared with the values obtained by previous investigators. In addition, another numerical scheme based on an integral-equation formulation is derived to compute solitary waves of arbitrary amplitude. These calculations confirm and extend the calculations of Byatt-Smith & Longuet-Higgins (1976) for very steep waves.

1. Introduction

Since the time of Russell (1845), many approximate solutions for solitary waves have been obtained. Solutions in the form of an expansion in powers of the wave amplitude were derived by Rayleigh (1876), Korteweg & de Vries (1895), Keller (1948), Laitone (1960), Fenton (1972), Longuet-Higgins & Fenton (1974), Witting (1975) and others. On the other hand, direct numerical calculations were attempted by Yamada (1957), Lenau (1966), Yamada, Kimura & Okabe (1968), Byatt-Smith (1971), Byatt-Smith & Longuet-Higgins (1976), Witting (1981) and Witting & Bergin (1981). A review of some of these investigations can be found in Miles (1980).

Most of these calculations are in good agreement for relatively small values of the wave height

$$\alpha = A/H. \tag{1.1}$$

Here A is the elevation of the crest of the wave, measured from the undisturbed level of the free surface, and H is the undisturbed depth.

However, some discrepancies appear as the wave of maximum height is approached. For example the following numerical values for the maximum amplitude $\alpha_{\max}$ have been obtained: 0.827 ± 0.008 (Yamada 1957), 0.827 (Lenau 1966), 0.8262 (Yamada *et al.* 1968), 0.827 (Longuet-Higgins & Fenton 1974), 0.8332 (Witting 1981; Witting & Bergin 1981) and 0.8332 (Fox (1978), unpublished dissertation mentioned by Schwartz & Fenton (1982) and by Longuet-Higgins (1980)).

In a recent paper Williams (1981) presented accurate computations for periodic gravity waves of maximum height. His algorithm could not explicitly compute solitary waves. However, he approximated a solitary wave by a long periodic wave, and obtained the value 0.833 197 for the maximum height.

Accurate solutions for steep solitary waves were obtained by Longuet-Higgins & Fenton (1974) and Byatt-Smith & Longuet-Higgins (1976). Both calculations predict that the highest solitary wave is not the fastest. However, the results predicted by these calculations do not agree for very steep waves (see figure 3). On the other hand, the results of Witting (1981) and Witting & Bergin (1981) agree with those of Byatt-Smith & Longuet-Higgins (1976).

In this paper we present a numerical scheme based on series truncation to compute the solitary wave of maximum height. The method is akin to that of Lenau (1966). However, our results are more accurate, since we retain up to 100 terms in the power expansion, whereas Lenau retained only 9 terms. It is found that $\alpha_{\max} = 0.83322$. This value is about 0.006 higher than the values obtained by Yamada (1957), Lenau (1966), Yamada *et al.* (1968) and Longuet-Higgins & Fenton (1974). On the other hand, it agrees to four places with the values obtained by Witting (1981), Witting & Bergin (1981) and Williams (1981). We also show that Yamada's (1957) scheme yields the value 0.833 when a sufficiently large number of mesh points is used. A similar result was found by Witting (1981) and Witting & Bergin (1981).

In addition, we present another numerical scheme based on an integral-equation formulation to compute solitary waves of arbitrary amplitude. The method is similar in philosophy, if not in details, to the scheme derived by Vanden-Broeck & Schwartz (1979).

Following Longuet-Higgins & Fenton (1974) we introduce the parameter

$$\omega = 1 - \frac{q_c^2}{gH}. \tag{1.2}$$

Here q_c is the velocity at the crest of the wave and g is the acceleration due to gravity. The parameter ω varies between 0 and 1 as the wave amplitude varies from zero to its maximum value.

The numerical solutions of our integral equation differ from the results of Longuet-Higgins & Fenton (1974) for $\omega > 0.92$. On the other hand, they agree with the numerical results of Byatt-Smith & Longuet-Higgins (1976) for $\omega \leqslant 0.96$. Byatt-Smith & Longuet-Higgins also used an integral-equation formulation. However, they were not able to compute waves for $\omega > 0.96$, because too many mesh points were required to describe accurately the flow in the neighbourhood of the crest. In the present work this difficulty is avoided by concentrating the mesh points near the crest by an appropriate change of variable. This enables us to compute accurate solutions up to $\omega = 0.99$. An extrapolation of these results shows that $\alpha \rightarrow 0.833$ as $\omega \rightarrow 1$. This constitutes an important check on the consistency of our two numerical schemes.

The problem is formulated in §2, and the highest wave is calculated in §3. In §4 we compute solitary waves of arbitrary amplitude via an integral-equation formulation. The results are discussed in §5.

2. Formulation

We consider a two-dimensional solitary wave in an inviscid incompressible and irrotational fluid, bounded below by a horizontal bottom. We take a frame of reference with the x-axis parallel to the bottom and moving with the phase velocity c of the wave. The level $y = 0$ is chosen as the undisturbed level of the free surface, and gravity is assumed to act in the negative y-direction.

We introduce the potential function $\phi(x, y)$ and the stream function $\psi(x, y)$. Without loss of generality, we choose $\phi = 0$ at the crest and $\psi = 0$ on the free surface.

We denote by Q the value of ψ on the bottom. Then the undisturbed depth H is given by

$$H = Q/c. \tag{2.1}$$

We introduce dimensionless variables by taking H as the unit length and c as the unit velocity. We choose the complex potential

$$f = \phi + \mathrm{i}\psi \tag{2.2}$$

as the independent variable.

We shall seek the complex velocity

$$\zeta = u - \mathrm{i}v \tag{2.3}$$

as an analytic function of f in the strip $-1 < \psi < 0$. At infinity we require the velocity to be c in the x-direction, so that the dimensionless velocity is unity in the x-direction. Therefore ζ must tend to unity at infinity.

On the free surface, the Bernoulli equation yields

$$\tfrac{1}{2}F^2[u^2(\phi)+v^2(\phi)] + \int_0^\phi \frac{v(s)}{u^2(s)+v^2(s)}\,\mathrm{d}s = \tfrac{1}{2}F^2 - \alpha \quad \text{on} \quad \psi = 0. \tag{2.4}$$

Here α is the elevation of the crest and F is the Froude number, defined by

$$F = \frac{c}{(gH)^{\frac{1}{2}}}. \tag{2.5}$$

The functions $u(\phi)$ and $v(\phi)$ in (2.4) denote respectively $u(\phi, 0_-)$ and $v(\phi, 0_-)$.

On the bottom, the kinematic boundary condition yields

$$v = 0 \quad \text{on} \quad \psi = -1. \tag{2.6}$$

This completes the formulation of the problem of determining the analytic function ζ. This function must tend to unity at infinity, satisfy (2.4) on $\psi = 0$, and (2.6) on $\psi = -1$.

Finally, let us mention that the asymptotic behaviour of $u(\phi) - \mathrm{i}v(\phi)$ as $\phi \to \pm\infty$ is described by Stokes' result

$$u(\phi) - \mathrm{i}v(\phi) \sim A\,\mathrm{e}^{-\pi\lambda|\phi|} \quad \text{as} \quad \phi \to \pm\infty. \tag{2.7}$$

Here A is a complex constant to be found as part of the solution, and λ is the smallest root of

$$\pi\lambda - \frac{\tan \pi\lambda}{F^2} = 0. \tag{2.8}$$

3. The highest solitary wave

In this section we present a numerical scheme based on series truncation to compute the highest solitary wave. This wave is characterized by a stagnation point at the crest, where the surface makes a 120° angle with itself (see figure 1). Following Lenau (1966), we introduce the new variable t by the relation

$$f = \frac{2}{\pi}\log\frac{1+t}{1-t} - \mathrm{i}. \tag{3.1}$$

This transformation maps the flow domain onto the domain $\{|t| < 1, \operatorname{Im} t > 0\}$ in the complex t-plane (see figure 2). The points $\phi = -\infty$ and $\phi = +\infty$ are mapped onto

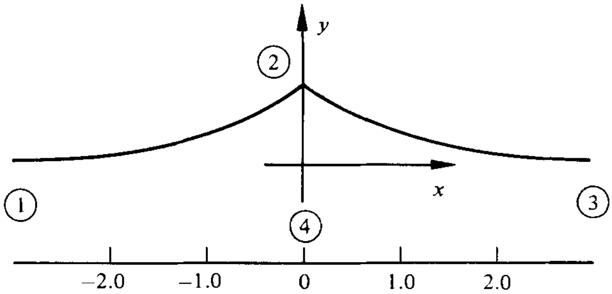

FIGURE 1. Computed free-surface profile for the highest solitary wave. The vertical scale is the same as the horizontal scale.

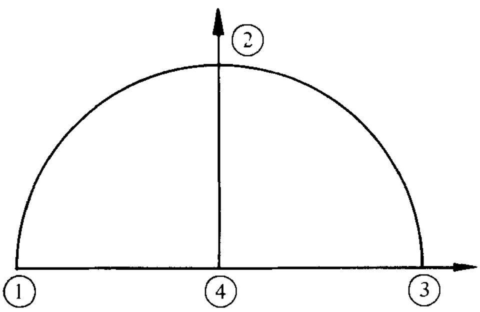

FIGURE 2. Flow configuration in the complex t-plane.

the points $t = -1$ and $t = +1$. These points are labelled by the numbers 1 and 3 in figures 1 and 2. The crest of the wave and the point $\phi = 0$, $\psi = -1$ are labelled by the numbers 2 and 4 in figure 1. They are mapped onto the points $t = \mathrm{i}$ and $t = 0$ in figure 2. We use the notation $t = r\,\mathrm{e}^{\mathrm{i}\sigma}$, so that the free surface is described by $r = 1$, $0 \leqslant \sigma \leqslant \pi$.

Lenau (1966) derived the following expansion for the complex velocity ζ:

$$\zeta = [\tfrac{1}{2}(1+t^2)]^{\frac{1}{3}}\,\mathrm{e}^{\Omega(t)}, \tag{3.2}$$

where

$$\Omega(t) = A(1-t^2)^{2\lambda} + \sum_{n=0}^{\infty} a_{n+1}(t^{2n}-1). \tag{3.3}$$

Here λ is the smallest root of (2.8). The coefficients A and a_i ($i = 1, 2, 3, \ldots$) in (3.3) have to be found to satisfy the boundary condition (2.4).

We solve the problem approximately by truncating the infinite sum in (3.3) after N terms. Differentiating (2.4) with respect to σ and using (3.1) yields

$$F^2[\tilde{u}(\sigma)\,\tilde{u}_\sigma(\sigma) + \tilde{v}(\sigma)\,\tilde{v}_\sigma(\sigma)] - \frac{2}{\pi}\frac{\tilde{v}(\sigma)}{\tilde{u}^2(\sigma)+\tilde{u}^2(\sigma)}\frac{1}{\sin\sigma} = 0. \tag{3.4}$$

Here $\tilde{u}(\sigma) = u[\phi(\sigma)]$ and $\tilde{v}(\sigma) = v[\phi(\sigma)]$ are the components of the velocity on the free surface.

The functions $\tilde{u}$ and $\tilde{v}$, and their derivatives $\tilde{u}_\sigma$ and $\tilde{v}_\sigma$, are obtained in terms of A, λ and a_i ($i = 1, \ldots, N$) by substituting $t = \mathrm{e}^{\mathrm{i}\sigma}$ in (3.2). We find the $N+3$ unknowns λ, A, F, a_i ($i = 1, \ldots, N$) by satisfying (3.4) at the $N+2$ mesh points

$$\sigma_I = \frac{\pi}{2(N+2)}\left(I - \frac{1}{2}\right) \quad (I = 1, \ldots, N+2). \tag{3.5}$$

N	F	N	F
9	1.28998	50	1.29089
15	1.29055	75	1.29091
30	1.29083	100	1.29091

TABLE 1. Values of the Froude number of the highest wave for various values of N

Thus we obtain a system of $N+2$ nonlinear algebraic equations. The last equation is obtained by imposing (2.8).

This system of $N+3$ equations for the $N+3$ unknowns λ, A, F, a_i $(i = 1, \ldots, N)$ was solved by Newton iterations. For most calculations the values $a_1 = -\frac{1}{6}$, $a_i = 0$ $(i = 2, \ldots, N)$, $A = -0.32$ and $F = 1.3$ were used as the initial guess. The method converges rapidly and a residual error of 10^{-10} was obtained after 4 or 5 iterations. Furthermore, the coefficients a_i decrease rapidly as i increases. For example $a_{30} \sim 4\times 10^{-5}$, $a_{60} \sim 6\times 10^{-6}$, $a_{100} \sim 10^{-8}$. It is interesting to note that good convergence was obtained without including Grant's (1973) singularity in the expansion. As a check on our scheme, we plotted the values of $\log\,[z-(\frac{3}{2}F\phi\,\mathrm{e}^{-\mathrm{i}\pi/4})^{\frac{2}{3}}-\frac{1}{2}\mathrm{i}F^2]$ versus $\log\phi$ for various points on the free surface. The plotted points followed very closely a straight line of slope 1.5. This result is in agreement with Grant's (1973) expansion.

Numerical values of F for various values of N are shown in table 1. These results indicate that the value $F = 1.29091$ is correct to 5 decimal places. The profile of the wave is shown in figure 1.

The highest solitary wave is characterized by $u = v = 0$ at the crest $\phi = \psi = 0$. Therefore (2.4) shows that the amplitude $\alpha_{\max}$ of the highest wave is given by

$$\alpha_{\max} = \tfrac{1}{2}F^2 = 0.83322. \tag{3.6}$$

This value is about 0.006 higher than the values obtained by Yamada (1957), Lenau (1966), Yamada *et al.* (1968) and Longuet-Higgins & Fenton (1974). On the other hand, it agrees to four places with the values obtained by Witting (1981), Witting & Bergin (1981) and Williams (1981).

Our numerical method differs from that of Lenau (1966) because we satisfy (3.4) at the mesh points (3.5) instead of solving for the Fourier coefficients. It is also more accurate because we retain up to 100 terms in (3.3), whereas Lenau retained only 9 terms.

As a further check on our calculations we repeated Yamada's (1957) calculations. Yamada (1957) presented the value 0.827 ± 0.008 obtained with 11 mesh points. With 11 mesh points we also obtained 0.827. However, we obtain 0.832 with 30 mesh points and 0.833 with 100 mesh points. Thus Yamada's (1957) scheme yields the same answer correct to three figures when a sufficiently large number of mesh points is used.

4. Numerical solution via an integral equation

It is convenient to reformulate the problem as an integrodifferential equation by considering $u-\mathrm{i}v-1$. This function tends to zero at infinity. In order to satisfy the boundary condition (2.6) on $\psi = -1$, we reflect the flow in the boundary $\psi = -1$. Thus we seek $u-\mathrm{i}v-1$ as an analytic function of f in the strip $-2 \leqslant \psi \leqslant 0$.

The values of u and v on the free surface $\psi = 0$ and its image $\psi = -2$ are related by the identities

$$u(\phi, 0) = u(\phi, -2), \tag{4.1}$$

$$v(\phi, 0) = -v(\phi, -2). \tag{4.2}$$

In order to find a relation between $u(\phi, 0)$ and $v(\phi, 0)$, we apply Cauchy's theorem to the function $u - \mathrm{i}v - 1$ in the strip $-2 \leqslant \psi \leqslant 0$. Using (4.1) and (4.2), and exploiting the bilateral symmetry of the wave about $\phi = 0$, we obtain, after some algebra,

$$\begin{aligned} u(\phi) - 1 = {} & \frac{1}{\pi}\int_0^\infty v(s)\left[\frac{1}{s-\phi} + \frac{1}{s+\phi}\right]\mathrm{d}s \\ & + \frac{1}{\pi}\int_0^\infty \frac{(s-\phi)\,v(s) + 2[u(s)-1]}{(s-\phi)^2+4}\,\mathrm{d}s \\ & + \frac{1}{\pi}\int_0^\infty \frac{(s+\phi)\,v(s) + 2[u(s)-1]}{(s+\phi)^2+4}\,\mathrm{d}s. \end{aligned} \tag{4.3}$$

The first integral in (4.3) is of Cauchy principal-value form. We shall measure the amplitude of the wave by the parameter ω. Using the symmetry of the wave about $\phi = 0$, we rewrite (1.2) in the form

$$\omega = 1 - F^2[u(0)]^2. \tag{4.4}$$

Using (4.4) and (2.4), evaluated at $\phi = 0$, we obtain

$$\alpha = \tfrac{1}{2}F^2 + \tfrac{1}{2}(\omega - 1). \tag{4.5}$$

For a given value of ω, (2.4), (4.3) and (4.5) define a system of integral equations for $u(\phi)$, $v(\phi)$, α and F.

In order to solve these equations, we find it convenient to introduce the new variable β, instead of ϕ, by the relation

$$\phi = \beta^\gamma, \quad \gamma > 1. \tag{4.6}$$

Therefore we rewrite (2.4), (4.3) and (4.5) in terms of β, $u^*(\beta) = u[\phi(\beta)]$ and $v^*(\beta) = v[\phi(\beta)]$.

Next we introduce the M mesh points

$$\beta_I = (I-1)\,E \quad (I = 1, \ldots, M), \tag{4.7}$$

where E is the interval of discretization. The change of variable (4.6) is chosen because it concentrates the mesh points near the crest of the wave. For very steep waves the value of γ was taken as 3.

We shall satisfy (2.4) and (4.3) at the points $\beta_{I+\frac{1}{2}} = \frac{1}{2}(\beta_I + \beta_{I+1})$ $(I = 1, \ldots, M-1)$. Following Vanden-Broeck & Schwartz (1979), we obtain, after discretization, $2M-2$ nonlinear algebraic equations for the $2M+2$ unknowns α, F and $u^*(\beta_I)$, $v^*(\beta_I)$ $(I = 1, \ldots, M)$. Relations (4.4) and (4.5) provide two more equations. An extra equation is obtained by imposing the symmetry condition

$$v^*(\beta_1) = 0. \tag{4.8}$$

The last equation expresses $u^*(\beta_M)$ in terms of $u^*(\beta_M - 1)$ and $u^*(\beta_{M-2})$ by an extrapolation formula based on the asymptotic formula (2.7). The discretization of (2.4) and (4.3) follows closely the work of Vanden-Broeck & Schwartz (1979).

The system of $2M+2$ equations was solved by Newton iterations.

$\phi_{max} = 7$		$\phi_{max} = 10$	
N	F	N	F
100	1.29141	100	1.29395
120	1.29143	120	1.29152
150	1.29145	150	1.29145
180	1.29145	180	1.29145

TABLE 2. Values of F when $\omega = 0.98$ and $\gamma = 3$

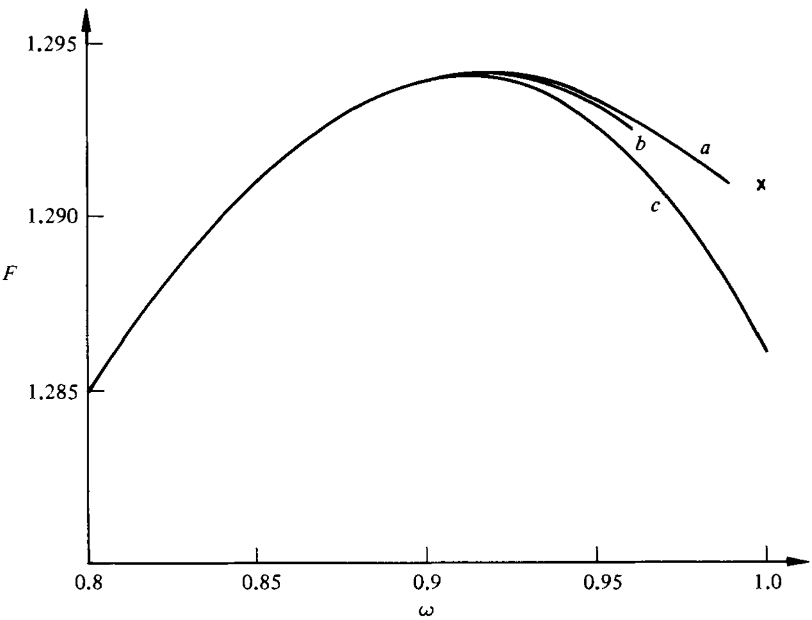

FIGURE 3. The Froude number F as a function of ω as given by the numerical scheme of §4 (curve a), Byatt-Smith & Longuet-Higgins (1976) (curve b) and Longuet-Higgins and Fenton (1974) (curve c). The cross corresponds to the highest wave calculated in §2.

The most important source of error in the numerical scheme arises from the truncation of the infinite integrals in (4.3) at

$$s = \phi_{max} = [(M-1)E]^{\gamma}. \tag{4.9}$$

We used two different methods to approximate the infinite integrals in (4.3). In the first method, we used the asymptotic formula (2.7) to approximate the integrals between ϕ_{max} and infinity. This approach is similar to the method used by Byatt-Smith & Longuet-Higgins (1976). In the second method, we simply neglected the contribution of the integrals between ϕ_{max} and infinity. In this second method we also replaced the equation in which an extrapolation based on (2.7) is used, by a Lagrange extrapolation formula. Thus, the second method is completely independent of (2.7). Both methods were found to give accurate results. However, the first method is more efficient because accurate results can be obtained with ϕ_{max} relatively small. Most of the results presented in the next section were obtained by using the first method.

5. Discussion of the results

In the first calculation the iterations were started with the classical solution of the Korteweg–de Vries equation. For ω small the iterations converged rapidly. Once a solution was obtained it was used as an initial guess for a larger value of ω, and so on.

For each value of ω we took E small enough and $\phi_{\max}$ large enough for the results to be independent of E and $\phi_{\max}$. This was achieved in the following way. For a given value of $\phi_{\max}$, we decreased E progressively to a value for which the results were independent of E, to the degree of accuracy desired. We repeated the procedure for larger and larger values of $\phi_{\max}$, up to a value for which the results were also independent of $\phi_{\max}$. This procedure is illustrated in table 2.

In figure 3 we present the numerical values of the Froude number F versus ω. These results confirm that the highest solitary wave is not the fastest. We also show the results obtained by Longuet-Higgins & Fenton (1974), and by Byatt-Smith & Longuet-Higgins (1976). Our results agree with those of Longuet-Higgins and Fenton for $\omega \leqslant 0.92$, and with those of Byatt-Smith & Longuet-Higgins for $\omega \leqslant 0.96$.

Byatt-Smith & Longuet-Higgins were not able to compute waves for $\omega > 0.96$ because their numerical procedure uses equal increments in the velocity potential. This is not well suited to the calculation of very steep waves, because large curvature, low velocity and sparse point spacing are characteristic of the crest region. In the present work this difficulty has been avoided by concentrating the mesh points near the crest, the change of variable (4.6).

Our results have thus confirmed the calculations of Byatt-Smith & Longuet-Higgins (1976) to at least three decimal places, and have extended them to still higher wave steepnesses, short of the maximum. For the limiting wave, our value for the wave height agrees with all the more recent calculations, to at least four decimal places, though disagreeing with Williams (1981) in the fifth.

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